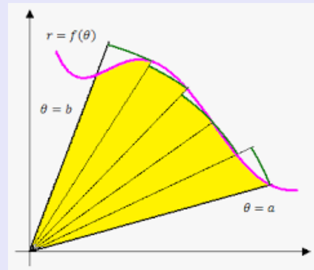
Calculus II

Lecture 18



Class QZ 15

Eliminate the parameter for the curve given by

$x = 5\cos t$, $y = 5\sin t$ for $0 \le t \le \pi$,

then graph it.

$\cos t = \frac{x}{5}$

$\sin t = \frac{y}{5}$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{5}\right)^2 + \left(\frac{x}{5}\right)^2 = 1$

$\to x^2 + y^2 = 25$

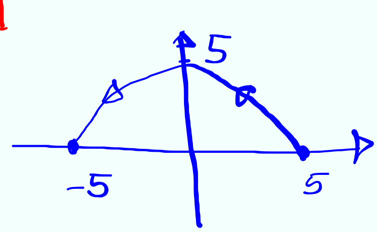
Circle

Center $(0,0)$, $r = 5$

$t = 0 \to x = 5$, $y = 0$

$\to t = \frac{\pi}{2}$ $x = 0$, $y = 5$

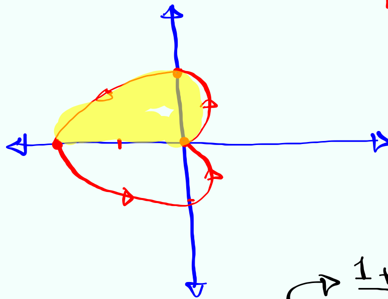
$t = \pi \to x = -5$, $y = 0$



Find the area enclosed by $r = 1 - \cos\theta$.

Polar Eqn

θ	r	
0	0	Pole
$\frac{\pi}{2}$	1	
π	2	
$\frac{3\pi}{2}$	1	
2π	0	



$\frac{1+\cos 2\theta}{2}$

$$A = \int_0^{2\pi} \frac{1}{2}\left[1-\cos\theta\right]^2 d\theta = 2\cdot\frac{1}{2}\int_0^{\pi}\left[1 - 2\cos\theta + \cos^2\theta\right]d\theta$$

$$= \left[\theta - 2\sin\theta + \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]\right]\Big|_0^{\pi}$$

$$= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi} = \boxed{\frac{3\pi}{2}}$$

Arc length in Polar

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \quad \text{in rectangular}$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{in parametric}$$

From the QZ

$x = 5\cos t$

$y = 5\sin t$

$0 \le t \le \pi$

$$L = \int_0^{\pi} \sqrt{(-5\sin t)^2 + (5\cos t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{25(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{\pi} 5\, dt = 5t\Big|_0^{\pi} = \boxed{5\pi}$$

$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$ in parametric

$$L=\int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

in polar

$x=r\cos\theta$

$y=r\sin\theta$

$$\frac{dx}{d\theta}=\frac{d}{d\theta}\left[r\cos\theta\right]=\frac{dr}{d\theta}\cdot\cos\theta-r\cdot\sin\theta$$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left[r\sin\theta\right]=\frac{dr}{d\theta}\sin\theta+r\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\overbrace{\left(\frac{dr}{d\theta}\cos\theta-r\sin\theta\right)^2}^{(A-B)^2}+\overbrace{\left(\frac{dr}{d\theta}\cdot\sin\theta+r\cos\theta\right)^2}^{(A+B)^2}$$

$$=\boxed{\left(\frac{dr}{d\theta}\right)^2\cos^2\theta}\cancel{-2r\sin\theta\cos\theta\frac{dr}{d\theta}}+\boxed{r^2\sin^2\theta}$$

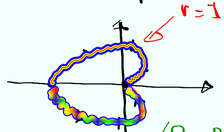
$$+\boxed{\left(\frac{dr}{d\theta}\right)^2\sin^2\theta}\cancel{+2r\sin\theta\cos\theta\frac{dr}{d\theta}}+\boxed{r^2\cos^2\theta}$$

$$=\boxed{\left(\frac{dr}{d\theta}\right)^2\underbrace{\left(\sin^2\theta+\cos^2\theta\right)}_{1}}+\boxed{r^2\underbrace{\left(\sin^2\theta+\cos^2\theta\right)}_{1}}$$

$$=r^2+\left(\frac{dr}{d\theta}\right)^2$$

$$\boxed{L=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}$$ Arc length in Polar

Find the arc length of the polar curve from the quiz



$$L=\int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2=\overbrace{(1-\cos\theta)^2}^{(A-B)^2}+(\sin\theta)^2$$

$$=1-2\cos\theta+\cos^2\theta+\sin^2\theta$$

$$=2(1-\cos\theta)$$

$$L=\int_0^{2\pi}\sqrt{2(1-\cos\theta)}\,d\theta=2\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\,d\theta$$

$$\int\sqrt{1-\cos\theta}\,d\theta=\int\sqrt{\frac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}}\,d\theta$$

$$=\int\frac{\sqrt{\sin^2\theta}}{\sqrt{1+\cos\theta}}\,d\theta=\int\frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$$

$u=1+\cos\theta$

$du=-\sin\theta\,d\theta$

$$=\int\frac{-1}{\sqrt{u}}\,du=\int-u^{-1/2}\,du=-\frac{u^{1/2}}{1/2}=-2\sqrt{1+\cos\theta}$$

$$L=2\sqrt{2}\left(-2\sqrt{1+\cos\theta}\right)\Big|_0^{\pi}=-4\sqrt{2}\left[0-\sqrt{2}\right]=\boxed{8}$$

Find the exact arc length of $r=2\cos\theta$ for $0\le\theta\le\pi$.

$r^2+\left(\frac{dr}{d\theta}\right)^2=$

$4\cos^2\theta+(-2\sin\theta)^2=$

$4\cos^2\theta+4\sin^2\theta=$

$4(\sin^2\theta+\cos^2\theta)=4\cdot 1=4$

$$L=\int_0^{\pi}\sqrt{4}\,d\theta$$

$$=\boxed{2\pi}\ ✓$$

$r=2\cos\theta$

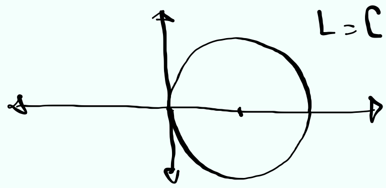
$r^2=2r\cos\theta$

$x^2+y^2=2x$

$x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1$

Circle center $(1,0)$, $r=1$

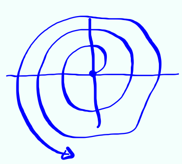


$L=C=2\pi r$

$=2\pi(1)$

$=2\pi$

Find the arc lengh of the polar Curve given by $r=\theta^2$ for $0\le\theta\le 2\pi$



$\frac{dr}{d\theta}=2\theta$

$r^2+\left(\frac{dr}{d\theta}\right)^2=\theta^4+4\theta^2$

$$L=\int_0^{2\pi}\sqrt{\theta^4+4\theta^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^2(\theta^2+4)}\,d\theta$$

$$=\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,d\theta$$

$u=\theta^2+4$

$du=2\theta\,d\theta$

$\frac{du}{2}=\theta\,d\theta$

$$=\int_4^{4\pi^2+4}\sqrt{u}\,\frac{du}{2}=\frac{1}{2}\left.\frac{u^{3/2}}{3/2}\right|_4^{4\pi^2+4}=\left.\frac{1}{3}u\sqrt{u}\right|_4^{4\pi^2+4}$$

$$=\frac{1}{3}\left[(4\pi^2+4)\sqrt{4\pi^2+4}-4\sqrt{4}\right]$$

$$=\frac{1}{3}\left[(8\pi^2+8)\sqrt{\pi^2+1}-8\right]=\frac{8}{3}\left[(\pi^2+1)^{3/2}-1\right]$$

Find the area of the region inside of both polar curves $r=\sqrt{3}\cos\theta$ & $r=\sin\theta$.

$r=\sqrt{3}\cos\theta$

$r^2=\sqrt{3}\, r\cos\theta$

$x^2+y^2=\sqrt{3}\, x$

$x^2-\sqrt{3}\,x+\left(\frac{\sqrt{3}}{2}\right)^2+y^2=\left(\frac{\sqrt{3}}{2}\right)^2$

$x^2-\sqrt{3}\,x+\frac{3}{4}+y^2=\frac{3}{4}$

$\left(x-\frac{\sqrt{3}}{2}\right)^2+(y-0)^2=\left(\frac{\sqrt{3}}{2}\right)^2$

center $\left(\frac{\sqrt{3}}{2},0\right)$

Radius $\frac{\sqrt{3}}{2}$

$r^2=r\sin\theta$

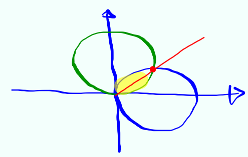
$x^2+y^2=y$

$x^2+y^2-y+\left(\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$

$(x-0)^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$

Circle

center $\left(0,\frac{1}{2}\right)$, $r=\frac{1}{2}$



Intersection pt

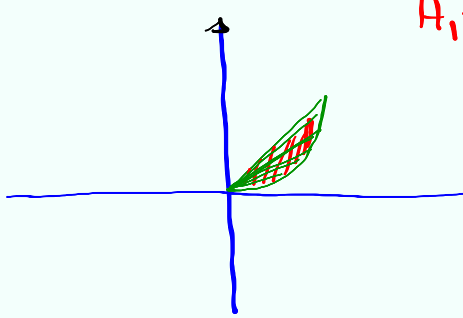
$\sin\theta=\sqrt{3}\cos\theta$

$\frac{\sin\theta}{\cos\theta}=\sqrt{3}$

$\tan\theta=\sqrt{3}$

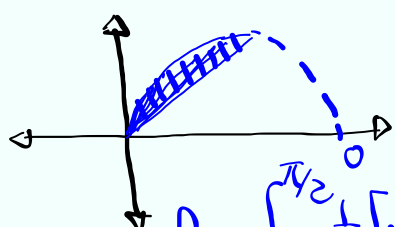
$\theta=\frac{\pi}{3}$

Green graph



$$A_1=\int_0^{\frac{\pi}{3}}\frac{1}{2}\left[\sin\theta\right]^2 d\theta$$

Blue graph



$A=A_1+A_2$

$$A_2=\int_{\pi/3}^{\pi/2}\frac{1}{2}\left[\sqrt{3}\cos\theta\right]^2 d\theta$$